

DIRECT SUM BEHAVIOR OF LATTICES OVER SIGMA- I RINGS

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This paper studies the lattice structure of a class of commutative rings called sigma- I (ΣI) rings, whose finitely generated torsionfree modules are direct sums of ideals. In particular, those ΣI rings that satisfy the Krull–Schmidt condition for lattices, those that satisfy cancellation and power cancellation and those that have the property that every lattice has a unique number of indecomposable direct summands are determined.

Introduction

A *ring-order* is a commutative, Noetherian, reduced ring R of Krull dimension 1 with module-finite integral closure \tilde{R} in its total quotient ring. A ring-order R is called ΣI if every *lattice*—i.e., a finitely generated submodule of a free module—is isomorphic to a direct sum of ideals of the ring. A *Bass ring* is a ring-order whose ideals are 2-generated; every Bass ring is ΣI [1, 7.3].

This paper is concerned with the following question: when are two direct sums of ideals of a ΣI ring R isomorphic R -modules? Steinitz [4] answered this question for Dedekind domains while Levy and Wiegand [3] provided the answer for Bass rings.

The Krull–Schmidt–Azumaya property (K - S - A)—i.e., that every lattice has a unique, up to isomorphism, decomposition into a direct sum of indecomposable lattices—holds for lattices over local ΣI rings (see Section 2). However, globally, quite a different situation prevails; see Section 4 for a complete description. Further, it is known that uniqueness of the number of indecomposable summands (denoted as $UNIS$)—i.e., that for each lattice M , there exists a unique integer $t = t(M)$ such that any indecomposable decomposition of M has t summands—can also fail over Bass rings. In Section 4, those ΣI rings that have the $UNIS$ property are completely characterized.

A major tool of the description of global direct sum behavior is the fact that the collection of isomorphism classes of lattices locally isomorphic to any given lattice M has a group structure $G(M)$, the *genus class group* of M . When R is a ΣI ring-order, every $G(M)$ is a homomorphic image of the direct product of some number t of copies of $G(R)$. For Bass rings, $t = 1$. I do not know whether $t = 1$

suffices for general ΣI rings. See Section 3. However, the existence of some t implies that the genus exponent and power-cancellation exponent are the same as they are for Bass rings. When $\text{Pic } R$ is finite:

(i) the genus exponent—the smallest e such that if M and N are locally isomorphic, then $M^{(e)}$ (direct sum of e copies) $\simeq N^{(e)}$ —equals the exponent of $\text{Pic } R$; and

(ii) the power-cancellation exponent—the smallest e such that $X \oplus M \simeq X \oplus N$ implies $M^{(e)} \simeq N^{(e)}$ —equals the exponent of a certain subgroup of $\text{Pic } R$. See Section 4.

1. Definitions and notations

Throughout this paper, R denotes a ring-order. If Q_1, \dots, Q_n denote the distinct minimal prime ideals of R , then the total quotient ring $K = Q(R)$ is a direct product of fields R_{Q_i} , while \tilde{R} is a Dedekind ring. The group of units of any ring S is denoted by S^* . If ${}_R M$ is a lattice, then the projective \tilde{R} -module, $\tilde{R} \otimes_R M / \text{torsion}$, is denoted by $\tilde{R}M$.

The conductor ideal $(R: \tilde{R}) = \{x \in \tilde{R} \mid x \cdot \tilde{R} \subset R\}$ is denoted by c . Since c contains a regular element of R , c is contained in only finitely many maximal ideals: $\mathcal{M}_1, \dots, \mathcal{M}_t$; these are called the *singular* maximal ideals of R .

In Sections 3 and 4, much of the notation of R. and S. Wiegand [5–7] will be used. In particular, given an R -lattice M and $u \in (\tilde{R}/c)^*$, M^u denotes the pullback of the following Cartesian square:

$$\begin{array}{ccc} M^u & \xrightarrow{\quad} & \tilde{R}M \\ \pi_1 \downarrow & & \downarrow \pi_3 \\ M/cM & \xrightarrow{\quad} & \tilde{R}M/cM \xrightarrow{\theta} \tilde{R}M/cM \end{array}$$

where θ is an automorphism of $\tilde{R}M/cM$ with determinant equal to u . The subgroup A_R of $(\tilde{R}/c)^*$ consists of elements that lift to units of \tilde{R} while the subgroup Δ_M consists of units which are determinants of automorphisms (over \tilde{R}/c) of $\tilde{R}M/cM$ that carry M/cM into itself.

2. Local direct sum behavior

This section identifies the indecomposable lattices and proves that the K-S-A condition holds for any local ΣI ring. Assume throughout this section that R is a local ΣI ring-order, which is necessarily either a local Bass ring or else a small class of ring-orders with precisely 3 minimal prime ideals [2].

Theorem 2.1. *Let R be a local Bass ring with integral closure \tilde{R} . Then*

(1) *There exist only finitely many rings S such that $R \subset S \subset \tilde{R}$, all of which form a chain of semilocal rings $R = R_0 \subset R_1 \subset R_2 \subset \cdots \subset R_n = \tilde{R}$ and are non-isomorphic as R -lattices; and*

(2) *Any indecomposable R -lattice X is isomorphic to either some ring $S = R_i$ or else R/Q for some minimal prime Q . Furthermore, every R_i is indecomposable except possibly \tilde{R} .*

Proof. (1) Most of these results follow immediately from [3, 2.1 and 2.2]. To see that each ring S between R and \tilde{R} is semilocal, note that $e \in \text{rad } S$ since any maximal ideal of S contains the unique maximal ideal of R . But S/e is Artinian so S has only finitely many maximal ideals.

(2) If X is a faithful indecomposable R -lattice, then by [3, 2.1], X is a projective ideal of the ring $S = \text{End}(X)$ where $R \subset S \subset \tilde{R}$. Since S is semilocal, it follows that $X \simeq S$.

If X is unfaithful, then $\text{Ann}_R(X) = Q$ is a minimal prime of R and, as Bass proves in [1; p. 22], R/Q is a discrete valuation ring (DVR). Since X is isomorphic to an ideal, $X \simeq R/Q$.

For the last statement, if R is a domain, then so is any ring S between R and \tilde{R} ; hence, S is indecomposable. If R is not a domain, then it is a subdirect sum of 2 discrete valuation domains R_1 and R_2 [1, 7.7]. Further, $\tilde{R} = R_1 \oplus R_2$ and any ring S properly contained in \tilde{R} contains no non-trivial idempotents; hence S is also indecomposable. \square

Definition 2.2. Let $f_i: R_i \rightarrow k$ ($i = 1, 2, 3$) be ring homomorphisms where each R_i is a DVR with maximal ideal $R_i \cdot x_i$ and k is a field. Define a *triad* of DVR's over a field to be the ring $R = \{(r_1, r_2, r_3) \in R_1 \oplus R_2 \oplus R_3 \mid f_1 r_1 = f_2 r_2 = f_3 r_3\}$. Let $S = \{(r_1, r_2) \in R_1 \oplus R_2 \mid f_1 r_1 = f_2 r_2\}$ and define $R' = \{(s, r_3) \in S \oplus R_3 \mid f_s s = g r_3\}$ where $f_s: S \rightarrow V$, $g: R_3 \rightarrow V$ are ring homomorphisms, V is an Artinian valuation ring of length 2, $\ker f_s = S \cdot (x_1, x_2)$ and $\ker g = R_3 \cdot x_3^2$. Then R' is called a *special quasi-triad* while the ring R is called the *associated triad* to the special quasi-triad R' . See [2, 3.1] for more details.

A local ΣI ring with exactly 3 minimal prime ideals must either be a triad or else a special quasi-triad [2, 3.2].

Theorem 2.3. *Let R be a local ΣI ring with 3 minimal primes Q_1, Q_2 and Q_3 . Then*

(1) *If R is a triad of the three DVR's R_1, R_2 and R_3 (so $R \subset R_1 \oplus R_2 \oplus R_3$) over the residue field k , then any indecomposable R -lattice X is isomorphic to one of the following: $R_i = R/Q_i$, $S_{ij} = R/(Q_i \cap Q_j)$, $I = R \cdot (1, 1, 0) + R \cdot (0, 1, 1)$ or R ; and*

(2) *If R is a special quasi-triad with T the associated triad for R , then any indecomposable R -lattice is isomorphic to either R or is an indecomposable T -lattice.*

Proof. (1) Let x_i generate the maximal ideal of the DVR $R_i = R/Q_i$. Identify R_i with its zero section in $R_1 \oplus R_2 \oplus R_3$ so that the maximal ideal of R becomes $\mathcal{M} = (x_1) \oplus (x_2) \oplus (x_3)$. Bass, in [1, p. 23], shows that indecomposable R -lattices are isomorphic to one of the following types:

$$(x_i) \simeq R/Q_i = R_i, \quad R/(x_k) \simeq R/(Q_i \cap Q_j) = S_{ij},$$

$$R \quad \text{and} \quad (x_1 + x_2, x_2 + x_3).$$

However, it is clear that $(x_1 + x_2, x_2 + x_3) \simeq R \cdot (1, 1, 0) + R \cdot (0, 1, 1) = I$.

(2) See [2, Theorem 3.10]. \square

Remark 2.4. An easy calculation shows that the endomorphism ring of the ideal I in Theorem 2.3 is the local ring R . Consequently, it is clear from inspection of Theorem 2.3 that every indecomposable R -lattice has a local endomorphism ring when R is either a triad or a special quasi-triad.

Theorem 2.5. *Let R be any local ΣI ring. Then R has the K-S-A property for lattices.*

Proof. Let M be an R -lattice and by passing from R to $R/\text{Ann}_R(M)$, assume that M is faithful.

If R is either a triad or a special quasi-triad of 3 DVR's, then by Remark 2.4, every indecomposable R -lattice has a local endomorphism ring and so the classical Krull–Schmidt–Azumaya theorem applies.

Assume now that R is a local Bass ring. If M has a decomposition with an unfaithful indecomposable summand, then since all such indecomposable lattices have local endomorphism rings by Theorem 2.1, any other decomposition must also have an isomorphic copy of that lattice as a summand. Furthermore, those summands can be cancelled. As a result, assume that any decomposition of M has only faithful indecomposable summands.

Using the results of Theorem 2.1, any two decompositions of M look like

$$R_0^{(a_0)} \oplus R_1^{(a_1)} \oplus \cdots \oplus R_n^{(a_n)} \simeq M \simeq R_0^{(b_0)} \oplus R_1^{(b_1)} \oplus \cdots \oplus R_n^{(b_n)} \quad (1)$$

where $a_i, b_i \geq 0$ and $R = R_0 \subset R_1 \subset R_1 \subset R_2 \subset \cdots \subset R_n \subseteq \tilde{R}$ are the indecomposable rings between R and \tilde{R} . Let k be the largest integer such that $a_k > 0$. Using the linear ordering of the R_i , a direct computation shows that $\rho(M) =: \{x \in \tilde{R} \mid x \cdot M \subset M\} = R_k$. (Note that $\rho(M)$ is the largest subring of \tilde{R} for which M is a module.) Since $\rho(M)$ is an isomorphism invariant, $b_k > 0$ and $b_j = 0$ for $j > k$. Yet cancellation holds for local ring-orders so cancel one copy of R_k from both sides. This process obviously continues by induction and this proves the K-S-A property for local Bass rings. \square

3. The genus class group of a lattice over a ring-order

Assume R is a ring-order, not necessarily local. If M is an R -lattice, the *genus* of M , denoted by $G(M)$, is the set of all isomorphism classes of lattices N such that $M_P \simeq N_P$ for all $P \in \text{Maxspec}(R) = \{\text{the maximal ideals of } R\}$. As shall be seen, a group structure, called the *genus class group*, on $G(M)$ is easily defined (see Theorem 3.1 below) and, in a certain sense, this group is ‘controlled’ by the Picard group of R (see Main Theorem 3.7 below). In making $G(M)$ into an additive abelian group, it is sometimes necessary to distinguish between a lattice M and its isomorphism class $[M]$. The first two results are due to Levy in a private communication.

Theorem 3.1 (Existence of genus class groups). *Let M be a faithful R -lattice. Then the genus of M , $G(M)$, is an additive abelian group with addition defined by $[X] + [Y] = [S]$ where $[X], [Y], [S] \in G(M)$ and S satisfies $X \oplus Y \simeq M \oplus S$. Furthermore, the identity of $G(M)$ is the class $[M]$.*

Proof. Given $[X]$ and $[Y] \in G(M)$, the existence of the R -lattice S satisfying $X \oplus Y \simeq M \oplus S$ is due to a version of Roiter’s theorem that can be found in [5, 2.10]. But since cancellation holds for local rings, $S \in G(Y) = G(M)$. The operation is well defined for if $M \oplus S \simeq M \oplus T$, then by [6, 1.9], $S \simeq T^u$ for some $u \in \Delta_M$. But S, T , and M belong to the same genus so $\Delta_M = \Delta_S = \Delta_T$ (see [7, 2.3.2]). Hence $S \simeq T^u \simeq T$. The other group properties are easily checked. \square

Definition 3.2. Let $G = G(M)$ be the genus class group of a R -lattice M . The *restricted genus class group* of M , denoted by $D(M)$, is defined to be the set of lattices N in G such that $\tilde{R}N \simeq \tilde{R}M$. It is trivial to check that $D(M)$ forms a subgroup of G .

When M is a faithful R -lattice, an equivalent definition of $D(M)$ is sometimes more useful. Let $\text{Pic } \tilde{R} = G(\tilde{R})$ denote the Picard group of \tilde{R} , written additively, and define a map $\beta: G \rightarrow \text{Pic } \tilde{R}$ via $\beta([N]) = \text{cl}(\tilde{R}N) \cdot \text{cl}(\tilde{R}M)^{-1}$ where $\text{cl}(X)$ denotes the ideal class (in the Steinitz sense) in the Dedekind ring \tilde{R} . Using the definition of $G(M)$, it easily checked that β is a group homomorphism. Furthermore, by [5, 2.9], β is an epimorphism. In this case, the restricted genus class group is the kernel of the map β ; clearly, $D(M)$ is a subgroup of $G(M)$ and from [6, 1.4], $D(M) = \{[M^u] \mid u \in (\tilde{R}/c)^*\} \simeq (\tilde{R}/c)^* / \Delta_M \Lambda_R$.

The main result of this section is that every $G(M)$ is a homomorphic image of a direct product of t copies of $\text{Pic } R$, denoted $\prod^t \text{Pic } R$; in this sense, $\text{Pic } R$ ‘controls’ $G(M)$. Furthermore, the integer t depends only on R . In order to show this, a series of lemmas, showing there exists only finitely many ‘basic building blocks’ of genera and that there are several group homomorphisms between the various genera, is necessary.

Lemma 3.3. *Let R be a ΣI ring-order. Then there exist only finitely many genera of indecomposable R -lattices, $G(Y_1), \dots, G(Y_t)$; i.e., each Y_i is indecomposable.*

Proof. Since R is ΣI , it suffices to show there exist only finitely many genera of ideals of R . Set $S = R \setminus \{P \in \text{Spec } R \mid P \text{ is either a minimal prime ideal or a singular maximal ideal}\}$ and note that for any ideal $I \neq 0$ of R , $S^{-1}I \neq 0$ is an ideal of $S^{-1}R$. Further, since R_P is a local Dedekind domain whenever P is a non-singular maximal ideal, the genera of an R -ideal is determined solely by its localizations at the minimal prime and singular maximal ideals. Thus, there is a one-to-one correspondence between the class of genera of ideals I of R and the class of genera of ideals $S^{-1}I$ of $S^{-1}R$ given by $G(I) \leftrightarrow G(S^{-1}I)$. Without loss of generality, replace R by the semilocal ring $S^{-1}R$; in this case, the genus of a lattice is its isomorphism class.

Now R has only finitely many minimal prime ideals and only finitely many maximal ideals, each of which is singular. Further, two ideals are isomorphic if their localizations at each of those prime ideals are isomorphic. Since the only ideals of R_Q are R_Q and 0 whenever Q is a minimal prime ideal, it suffices to show that there are only finitely many ideals of R_P where P is a singular maximal ideal. But from Theorems 2.1 and 2.3, R_P has only finitely many non-isomorphic ideals. \square

Lemma 3.4. *Let M and N be R -lattices belonging to the same genus. Then there exist group isomorphisms $\theta: G(M) \xrightarrow{\sim} G(M \oplus N)$ and $\theta|_{D(M)}: D(M) \xrightarrow{\sim} D(M \oplus N)$ defined by $\theta([X]) = [X \oplus N]$.*

Proof. By replacing R by $R/\text{Ann}_R(M)$, if necessary, assume M and N are faithful R -lattices and define θ as above. Using the definitions of $G(M)$ and $D(M)$, the reader can check that θ is a group homomorphism. Note $\theta(D(M)) \subset D(M \oplus N)$ since $\theta([M^u]) = [M^u \oplus N] = [(M \oplus N)^u] \in D(M \oplus N)$ by [5, 2.2].

It is straightforward to check, by diagram chasing, that the following diagram is commutative:

$$\begin{array}{ccccc} D(M) & \xrightarrow{\quad} & G(M) & \xrightarrow{\beta} & \text{Pic}(\tilde{R}) \\ \theta \downarrow & & \theta \downarrow & & \downarrow = \\ D(M \oplus N) & \xrightarrow{\quad} & G(M \oplus N) & \xrightarrow{\beta} & \text{Pic}(\tilde{R}) \end{array}$$

As a result, to show that θ is an isomorphism, it suffices, by the Five Lemma, to show that $\theta|_{D(M)}$ is an isomorphism. Yet $\theta([M^u]) = [M^u \oplus N] = [(M \oplus N)^u]$ so, by Definition 3.2, $\theta|_{D(M)}$ is epic. To see that $\theta|_{D(M)}$ is one-to-one, note that if $[M^u] \in \text{kernel}$, then $\theta([M^u]) = [M^u \oplus N] = [M \oplus N]$. So by [6, 1.6], $u \in \Delta_{M \oplus N}$. Yet M and N are in the same genus so by [7, 2.1.4 and 2.3.2], $\Delta_{M \oplus N} = \Delta_M \Delta_N = \Delta_M \cdot \Delta_M = \Delta_M$. Hence, $[M^u] = [M]$ or $\ker \theta|_{D(M)} = 0$. \square

Lemma 3.5. *Let M and N be R -lattices. Then there exist group epimorphisms $\varphi: G(M) \oplus G(N) \rightarrow G(M \oplus N)$ and $\varphi|_{D(M) \oplus D(N)}: D(M) \oplus D(N) \rightarrow D(M \oplus N)$ defined via $\varphi([X], [Y]) = [X \oplus Y]$.*

Proof. By passing to $R/\text{Ann}(M \oplus N)$, assume $M \oplus N$ is faithful. Since M and N may not be faithful and their annihilators could be different, recall that \tilde{R} is a direct sum of Dedekind domains and that, from [5, 2.2], the property $M^u \oplus N^v \simeq (M \oplus N)^{uv}$ holds provided $u \in (\tilde{R}/c)^*$ (respectively, $v \in (\tilde{R}/c)^*$) has all coordinates equal to 1 in any component which annihilates M (respectively, N). In this sense, $\varphi([M^u], [N^v]) = [M^u \oplus N^v] = [(M \oplus N)^{uv}]$ and so $\varphi(D(M) \oplus D(N)) \subset D(M \oplus N)$. (I thank the referee for bringing this point to my attention.)

Now, as in Lemma 3.4, it is easily checked that φ is a group homomorphism and that the following diagram is commutative:

$$\begin{array}{ccccc} D(M) \oplus D(N) & \xrightarrow{\quad} & G(M) \oplus G(N) & \xrightarrow{\beta \oplus \beta} & \text{Pic}(\tilde{R}) \oplus \text{Pic}(\tilde{R}) \\ \varphi \downarrow & & \varphi \downarrow & & \downarrow \\ D(M \oplus N) & \xrightarrow{\quad} & G(M \oplus N) & \xrightarrow{\beta} & \text{Pic}(\tilde{R}) \end{array}$$

where the right vertical map sends $(X, Y) \rightarrow (XY)$. Once again, to see that φ is epic, it suffices to show $\varphi|_{D(M) \oplus D(N)}$ is epic. But every unit x in $(\tilde{R}/c)^*$ can be written as a product $u \cdot v$, where u and v are as in the remark in the above paragraph. As a result,

$$\varphi([M^u], [N^v]) = [(M \oplus N)^{uv}] = [(M \oplus N)^x]$$

and so $\varphi|_{D(M) \oplus D(N)}$ is epic. \square

Lemma 3.6. *Let I be an ideal of R . Then there exists epimorphisms $\sigma: \text{Pic } R \rightarrow G(I)$ and $\sigma|_{D(R)}: D(R) \rightarrow D(I)$ defined by $\sigma([H]) = [H \cdot I]$ where H is an invertible R -fractional ideal in \tilde{R} and $H \cdot I$ is the lattice formed by ordinary multiplication within $\tilde{R}I$.*

Proof. By replacing R by $R/\text{Ann}_R(I)$, assume that I is faithful and define the map $\sigma: \text{Pic } R \rightarrow G(I)$ as above. Again, it is straightforward to check that σ is a well-defined homomorphism. Note that in $\text{Pic } R = G(R)$, $[H] + [K] = [H \cdot K]$.

Claim. $R^u I = I^u$ and so $\sigma(D(R)) \subset D(I)$.

To see this, note that, by [5, 2.2], $R^u \oplus I \simeq (R \oplus I)^u \simeq R \oplus I^u$ and so applying the class ideal functor, as defined in [3, 3.2], $R^u \cdot I = I^u$ as desired.

As a result of the above claim, the following diagram is clearly commutative:

$$\begin{array}{ccccccc} 0 & \longrightarrow & D(R) & \longrightarrow & \text{Pic } R & \xrightarrow{\beta} & \text{Pic } \tilde{R} \longrightarrow 0 \\ & & \sigma \downarrow & & \sigma \downarrow & & \downarrow = \\ 0 & \longrightarrow & D(I) & \longrightarrow & G(I) & \xrightarrow{\beta} & \text{Pic } \tilde{R} \longrightarrow 0 \end{array}$$

The fact that $R^u I = I^u$ also shows that $\sigma: D(R) \rightarrow D(I)$ is onto and so by the Snake Lemma, $\sigma: \text{Pic } R \rightarrow G(I)$ must be onto as well. \square

Main Theorem 3.7. *Let R be a ΣI ring with genera $G(Y_1), \dots, G(Y_t)$ as in Lemma 3.3. Let L be any R -lattice. Then $G(L)$ [respectively, $D(L)$] is a homomorphic image of $\prod^t \text{Pic } R$ (respectively, $\prod^t D(R)$).*

Proof. Since L decomposes into a direct sum of indecomposable lattices, each of which belongs to one of $G(Y_1), \dots, G(Y_t)$, it follows by suitable grouping that $G(L) = G(Y_{j_1}^{(k_1)} \oplus Y_{j_2}^{(k_2)} \oplus \dots \oplus Y_{j_s}^{(k_s)})$. From Lemma 3.4, it follows that $G(L) \simeq G(Y_{j_1} \oplus Y_{j_2} \oplus \dots \oplus Y_{j_s})$. But using Lemmas 3.5 and 3.6, the following epimorphisms exist:

$$\begin{aligned} \prod^t \text{Pic } R &\rightarrow G(Y_1 \oplus Y_2 \oplus \dots \oplus Y_t) \\ &\rightarrow G(Y_{j_1} \oplus Y_{j_2} \oplus \dots \oplus Y_{j_s}) \simeq G(L). \end{aligned}$$

In a similar vein, the epimorphisms

$$\begin{aligned} \prod^t D(R) &\rightarrow D(Y_1 \oplus Y_2 \oplus \dots \oplus Y_t) \\ &\rightarrow D(Y_{j_1} \oplus Y_{j_2} \oplus \dots \oplus Y_{j_s}) \simeq D(L) \end{aligned}$$

also exist. \square

4. Global direct sum behavior

Section 2 showed that local ΣI rings satisfy the K-S-A condition for lattices. For the global situation, K-S-A holds only when two very strong conditions occur.

Lemma 4.1. *Let R be an indecomposable ΣI ring with exactly 1 singular maximal ideal \mathcal{M} and let $X \neq 0$ be an R -lattice. Then X is indecomposable $\Leftrightarrow X_{\mathcal{M}}$ is indecomposable.*

Proof ‘ \Rightarrow ’. Suppose $X_{\mathcal{M}} = L \oplus N$ where L and N are $R_{\mathcal{M}}$ -lattices. Let θ' be a projection of $X_{\mathcal{M}}$ onto L so that θ' is an idempotent endomorphism of $X_{\mathcal{M}}$. But $\text{End}_{R_{\mathcal{M}}}(X_{\mathcal{M}}) \simeq R_{\mathcal{M}} \otimes_R \text{End}_R(X)$ and so $\theta' = \theta/t$ for some $t \notin \mathcal{M}$ and $\theta \in \text{End}(X)$. Yet $\theta: X \rightarrow \theta(X)$ splits when localized at \mathcal{M} and at every non-singular maximal ideal P since R_P is a local Dedekind domain. Thus, θ splits at every maximal ideal of R and so X is decomposable.

‘ \Leftarrow ’. Suppose $X = Y \oplus Z$ where Y and Z are R -lattices. Since R is indecomposable, every minimal prime ideal is contained in \mathcal{M} . Thus, $Y \subset Y_{\mathcal{M}}$, $Z \subset Z_{\mathcal{M}}$ and $X \subset X_{\mathcal{M}}$ so that $X_{\mathcal{M}} = Y_{\mathcal{M}} \oplus Z_{\mathcal{M}}$ decomposes. \square

Theorem 4.2. *Let R be an indecomposable ΣI ring. Then R has the K-S-A property for lattices iff*

- (i) R has at most one singular maximal ideal, and
- (ii) $\text{Pic } R = 1$.

Proof. ‘ \Rightarrow ’. (i) Let \mathcal{M}_1 and \mathcal{M}_2 be distinct singular maximal ideals of R . It follows that $\mathcal{M}_i \neq R$ for otherwise $R_{\mathcal{M}_i}$ would be a local Dedekind domain (and hence would be non-singular). The epimorphism $\varphi: \mathcal{M}_1 \oplus \mathcal{M}_2 \twoheadrightarrow \mathcal{M}_1 + \mathcal{M}_2 = R$, defined via $\varphi(m_1, m_2) = m_1 - m_2$, splits (since R is projective) so that $R \oplus (\mathcal{M}_1 \cap \mathcal{M}_2) \simeq \mathcal{M}_1 \oplus \mathcal{M}_2$. But no indecomposable summand of \mathcal{M}_i can be isomorphic to R since $\mathcal{M}_i \subset R$ and $\mathcal{M}_i \neq R$. This isomorphism violates the K-S-A condition.

(ii) Suppose that $\text{Pic } R \neq 1$. Choose $I \neq R$ in $\text{Pic } R$ and observe that $I \oplus I^{-1} \simeq R \oplus I \cdot I^{-1} \simeq R \oplus R$. Thus, K-S-A fails if $\text{Pic } R \neq 1$.

‘ \Leftarrow ’. If R has no singular maximal ideals, then $R_{\mathcal{P}}$ is a DVR for every maximal ideal; it follows that R is a Dedekind domain with $\text{Pic } R = 1$ and so R has the K-S-A property. Assume that R has exactly one singular maximal ideal \mathcal{M} . Let X and Y be R -lattices.

Claim. If $X_{\mathcal{M}} \simeq Y_{\mathcal{M}}$, then $X \simeq Y$.

By Definition 3.2, if $\text{Pic } R = 1$, then $\text{Pic } \tilde{R} = 1$. Consequently, $\tilde{R}X \simeq \tilde{R}Y$ and $X \in G(Y)$ and so $X \simeq Y^u$ for some $u \in (\tilde{R}/c)^*$. However, $\text{Pic } R = 1$ implies $D(R) = 1$ and so by Main Theorem 3.7, $D(Y) = 0$. Consequently, $X \simeq Y^u \simeq Y$.

Now if an R -lattice M has decompositions

$$X_1 \oplus X_2 \oplus \cdots \oplus X_s \simeq Y_1 \oplus Y_2 \oplus \cdots \oplus Y_t,$$

then by localizing at \mathcal{M} , $M_{\mathcal{M}}$ decomposes as

$$(X_1)_{\mathcal{M}} \oplus (X_2)_{\mathcal{M}} \oplus \cdots \oplus (X_s)_{\mathcal{M}} \simeq (Y_1)_{\mathcal{M}} \oplus (Y_2)_{\mathcal{M}} \oplus \cdots \oplus (Y_t)_{\mathcal{M}}$$

where each $(X_i)_{\mathcal{M}}$ and each $(Y_j)_{\mathcal{M}}$ is indecomposable (by Lemma 4.1) and satisfies the claim. Yet Theorem 2.5 shows that local ΣI rings, such as $R_{\mathcal{M}}$, satisfy the K-S-A property. Therefore, $s = t$ and after relabeling, $(X_i)_{\mathcal{M}} \simeq (Y_i)_{\mathcal{M}}$ for all i . Applying the claim yields $X_i \simeq Y_i$ for all i and $s = t$. This proves that R satisfies the K-S-A property. \square

The question of which ΣI rings R have the UNIS property is a semi-local question. Consequently, the answer lies in the notion of the *graph* of the spectrum of R , which has the prime ideals of R as vertices and a directed edge connects vertex \mathcal{M} with vertex \mathcal{P} whenever $\mathcal{M} \supset \mathcal{P}$. The graph is denoted by $\text{gph}(R)$ and a *loop* is a simple closed path in $\text{gph}(R)$.

Another important tool is the notion of consistency as used heavily by Haefner and Levy in [2, 1.5]. For each maximal ideal \mathcal{M} of R , let $L(\mathcal{M})$ be an $R_{\mathcal{M}}$ -lattice and call a family of local lattices $\{L(\mathcal{M})\}$ *consistent* provided $L(\mathcal{M})_{\mathcal{Q}} \simeq L(\mathcal{M}')_{\mathcal{Q}}$

whenever Q is a minimal prime ideal contained in both \mathcal{M} and \mathcal{M}' . The following theorem is a method of glueing local lattices together and it follows by a slight generalization of [2, 1.6]:

Theorem 4.3 (Consistency Theorem). *Let $\{L(\mathcal{M})\}$ be a consistent family of local lattices. Then there exists an R -lattice L such that $L_{\mathcal{M}} \simeq L(\mathcal{M})$ for all maximal ideals \mathcal{M} of R . \square*

Lemma 4.4. *Let R be an indecomposable ring-order and let $\mathcal{M} = \{P \in \text{Spec } R \mid P \text{ contains at least 2 distinct minimal primes of } R\}$. Set $T = R \setminus \bigcup \{P \mid P \in \mathcal{M}\}$ and let $X \neq 0$ be an indecomposable R -lattice isomorphic to an ideal of R . Then $T^{-1}X$ is a non-zero, indecomposable $T^{-1}R$ -lattice isomorphic to an ideal of $T^{-1}R$.*

Proof. Since R is indecomposable, every minimal prime ideal is contained in some $P \in \mathcal{M}$. Thus, T consists of regular elements so $X \subset T^{-1}X \neq 0$.

Suppose that $T^{-1}X = Y \oplus Z$ where Y and Z are $T^{-1}R$ -lattices. Since X is isomorphic to an ideal of R , then for any minimal prime ideal Q , either $Y_Q = 0$ and $Z_Q = X_Q$ or vice versa. Furthermore, if P is a non-singular maximal ideal, then X_P is either an indecomposable ideal of the domain R_P or else is 0.

For each maximal ideal P of R , set $L(P) =: Y_P$ if $P \in \mathcal{M}$, $=: X_P$ if $P \notin \mathcal{M}$ and P contains some minimal prime Q where $Y_Q = X_Q \neq 0$, and $=: 0$ otherwise (i.e., if $Y_Q = 0$.) Similarly, set $N(P) =: Z_P$ if $P \in \mathcal{M}$, $=: X_P$ if $P \notin \mathcal{M}$ and P contains some minimal prime Q where $Z_Q = X_Q \neq 0$, and $=: 0$ otherwise (i.e., if $Z_Q = 0$.)

To see that the set $\{L(P) \mid P \in \text{Spec } R\}$ is consistent, let Q be a minimal prime such that $Q \subset P \cap P'$ where $P \in \mathcal{M}$ and $P' \notin \mathcal{M}$ and $X_Q \simeq Y_Q \neq 0$. Then $L(P)_Q = Y_Q \simeq X_Q = L(P')_Q$. If $P' \notin \mathcal{M}$ such that $Y_Q = 0$, then $L(P)_Q = Y_Q = 0 = L(P')_Q$.

In a similar manner, the set $\{N(P)\}$ is consistent. Furthermore, for each maximal ideal P , $X_P = L(P) \oplus N(P)$ and so by the Consistency Theorem, there exist lattices L and N such that $X \simeq L \oplus N$. Since X is indecomposable, either $L = 0$ or $N = 0$; this forces $Y = 0$ or $Z = 0$ and so $T^{-1}X$ is indecomposable. \square

Define the following three properties that R might have:

- (A) If X is any indecomposable R -lattice, then, for each maximal ideal P , $X_P = 0$ or is an indecomposable R_P -lattice.
- (B) The graph of R has no loops.
- (C) Every R -lattice M has a unique number of indecomposable summands.

Lemma 4.5. *Suppose R is an indecomposable ΣI ring. If $T^{-1}R$ has any one of the properties (A), (B) or (C), then R does as well.*

Proof. (a) Suppose $T^{-1}R$ has property (A) and let $X \neq 0$ be an indecomposable R -lattice. By Lemma 4.4, $T^{-1}X$ is an indecomposable $T^{-1}R$ -lattice, so $X_P \simeq (T^{-1}X)_P$ is indecomposable for all $P \in \mathcal{M}$ by (A). In addition, for $P \notin \mathcal{M}$, X_P is

indecomposable since R_P is a domain. Thus, X_P is indecomposable for all maximal ideals P of R .

(b) Suppose $T^{-1}R$ has property (B). By the definition of \mathcal{M} , any $P \notin \mathcal{M}$ contains exactly one minimal prime ideal of R and so P cannot contribute to a loop in $\text{gph}(R)$. Thus, since $\text{gph}(T^{-1}R)$ has no loops, neither does $\text{gph}(R)$.

(c) Suppose $T^{-1}R$ has property (C) and let M be any R -lattice with decompositions into indecomposable R -lattices as

$$M_1 \oplus \cdots \oplus M_s = M = N_1 \oplus \cdots \oplus N_t.$$

Then by localizing at T^{-1} ,

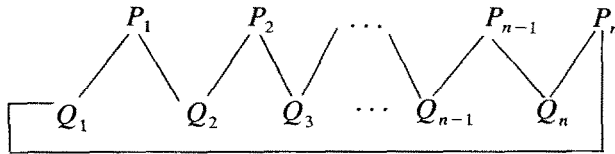
$$T^{-1}M_1 \oplus \cdots \oplus T^{-1}M_s = T^{-1}M = T^{-1}N_1 \oplus \cdots \oplus T^{-1}N_t,$$

where by Lemma 4.4, each $T^{-1}M_i$ and $T^{-1}N_j$ is non-zero and is indecomposable. But $T^{-1}R$ has property (C) and so $s = t$. \square

Theorem 4.6. *Let R be any ΣI ring. Then (A), (B) and (C) are equivalent properties.*

Proof. Without loss of generality, assume that R is an indecomposable ring. Furthermore, by Lemma 4.5, assume R is semi-local such that every maximal ideal contains at least two minimal prime ideals of R .

(A) \Rightarrow (B). Suppose the graph of R has a loop. Since R is ΣI , Haefner and Levy in [2, 1.2] show that the maximal ideals of the loop each must contain exactly two minimal prime ideals. Thus, part of the graph of R looks like



Construct an R -lattice X such that $X_P = R_P$ for all $P \neq P_n$ and $X_{P_n} = [R/Q_n]_{P_n} \oplus [R/Q_1]_{P_n}$ using the Consistency Theorem. (Consistency occurs because R and $R/Q_n \oplus R/Q_1$ have rank 1 at the minimal primes Q_1 and Q_n .)

To show that X is indecomposable, suppose $X = Y \oplus Z$ and suppose that $[R/Q_n]_{P_n}$ is a direct summand of Y_{P_n} . Clearly, $Y_{Q_n} \neq 0$ and so by the consistency of Y , $Y_{P_{n-1}} \neq 0$ as $Q_n \subset P_{n-1}$. Thus, by the indecomposability of $X_{P_{n-1}} = R_{P_{n-1}}$, $Y_{P_{n-1}} = X_{P_{n-1}} = R_{P_{n-1}}$ and $Z_{P_{n-1}} = 0$. But then it must be that $Y_{Q_{n-1}} \neq 0$ and so $Y_{P_{n-2}} \neq 0$. Continuing in this vein and using the diagram above, $Y_{Q_1} \neq 0$ and so $(Y_{P_n})_{Q_1} = (X_{P_n})_{Q_1} = ([R/Q_1]_{P_n})_{Q_1} \neq 0$. Hence $Y_{P_n} = X_{P_n}$ and $Y_P = X_P$ for $P \in \text{Spec } R$, so X is indecomposable. On the other hand, $X_{P_n} = [R/Q_n]_{P_n} \oplus [R/Q_1]_{P_n}$ decomposes so (A) is contradicted. This proves (A) \Rightarrow (B).

(B) \Rightarrow (C). Assume that the graph of R has no loops and let M be an arbitrary R -lattice. Let the minimal prime ideals of R be denoted by Q_1, Q_2, \dots, Q_n and recall that by assumption, every maximal ideal of R contains at least two minimal prime ideals.

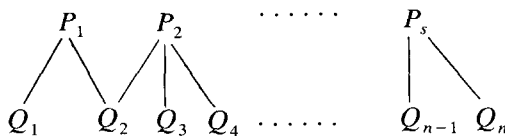
Claim. *There exists a maximal ideal P of R such that either*

(i) *P contains exactly two minimal primes Q_1, Q_2 and P is the only maximal ideal containing Q_1 , or*

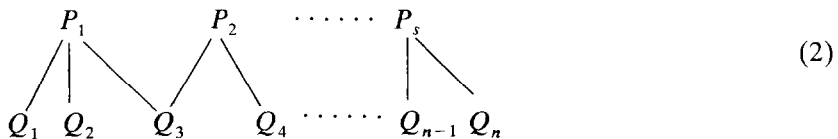
(ii) *P contains exactly three minimal primes Q_1, Q_2, Q_3 and P is the only maximal ideal containing either Q_1 or Q_2 .*

Define an *endpoint* of the graph of R to be a maximal ideal P satisfying either (i) or (ii) and call a minimal prime ideal Q a *terminal node*, provided it is contained in precisely one maximal ideal.

Proof of claim. Since the graph of R has no loops and since there are finitely many prime ideals, the $\text{gph}(R)$ must basically be an open path; i.e., there must be some endpoints to the graph. As a result, relabeling vertices if necessary, it must look like either:



or



In either case, the maximal ideals P_1 and P_s are endpoints of the graph. \square

To show (C), argue by induction on the number s of maximal ideals of R . If $s = 1$, then R , a local ΣI ring, has the K-S-A property by Theorem 2.5 which, of course, implies (C).

Now suppose that $s > 1$ and let P be an endpoint of the graph of R . Since both (i) and (ii) of the endpoint definition are treated similarly and since (ii) is more complex, assume P satisfies (ii). Thus,

P contains minimal prime ideals Q_1, Q_2 and Q_3 such that
 P is the only maximal ideal containing either Q_1 or Q_2 .

(3)

Diagram (2) above is applicable for this situation with $P = P_1$.

Since R is semi-local, we have by Lemma 3.3 that R has finite representation type; i.e., there exist only finitely many non-isomorphic indecomposable R -lattices X_1, \dots, X_t . As a result, one decomposition of M might be

$$M \simeq X_1^{(a_1)} \oplus \dots \oplus X_t^{(a_t)} \quad (4)$$

while another decomposition of M might be

$$M \simeq X_1^{(b_1)} \oplus \dots \oplus X_t^{(b_t)} . \quad (5)$$

Thus, to show (C), it suffices to show

$$\sum_{i=1}^t a_i = \sum_{i=1}^t b_i . \quad (6)$$

By (3), R/Q_1 , R/Q_2 and $R/(Q_1 \cap Q_2)$ are all local rings and so indecomposable R -lattices. Without loss of generality, set

$$X_1 =: R/Q_1, \quad X_2 =: R/Q_2 \quad \text{and} \quad X_3 =: R/(Q_1 \cap Q_2) . \quad (7)$$

Since these lattices are also rings, their endomorphism rings are also local. As a result, X_1 , X_2 and X_3 can be cancelled from the decompositions of M given in (4) and (5). This forces $a_1 = b_1$, $a_2 = b_2$ and $a_3 = b_3$ and so (4) and (5) reduce to

$$M' = X_4^{(a_4)} \oplus \dots \oplus X_t^{(a_t)} \simeq X_4^{(b_4)} \oplus \dots \oplus X_t^{(b_t)} . \quad (8)$$

Now it suffices to show that

$$\sum_{i=4}^t a_i = \sum_{i=4}^t b_i . \quad (9)$$

Form the set $S = R \setminus \{\mathcal{M} \in \text{Maxspec } R \mid \mathcal{M} \neq P\}$ which is not equal to R since $s > 1$. The crux of the inductive step is embedded in the next two claims. For $i > 3$,

$$S^{-1}X_i \neq 0 \quad (10)$$

and

$$S^{-1}X_i \text{ is an indecomposable } S^{-1}R\text{-lattice} . \quad (11)$$

To see that (10) is true, suppose that $S^{-1}X_i = 0$ for some $i > 3$. Then for all maximal ideals $\mathcal{M} \neq P$, $(X_i)_{\mathcal{M}} = 0$ and so in particular for all j such that $3 \leq j \leq n$, $(X_i)_{Q_j} = 0$. (See diagram (2); every such $Q_j \subset P_k$ for $k \geq 2$.) But this implies that $(X_i)_P = 0$, $(X_1)_P$, $(X_2)_P$ or $(X_3)_P$ and so $X_i \simeq 0$, X_1 , X_2 or X_3 since they agree locally. This contradicts the fact that $i > 3$ and so (10) is true.

To prove (11), suppose $S^{-1}X_i = Y \oplus Z$. Define, for every maximal ideal \mathcal{M} of R ,

$$\begin{aligned} L(\mathcal{M}) &= Y_{\mathcal{M}} \quad \text{for } \mathcal{M} \neq P, \\ L(P) &= \begin{cases} (X_i)_P & \text{if } Y_{Q_3} = (X_i)_{Q_3} \neq 0, \text{ and} \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Similarly, set

$$\begin{aligned} N(\mathcal{M}) &= Z_{\mathcal{M}} \quad \text{for } \mathcal{M} \neq P, \\ N(P) &= \begin{cases} (X_i)_P & \text{if } Z_{Q_3} = (X_i)_{Q_3} \neq 0, \text{ and} \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

To prove that the family $\{L(\mathcal{M})\}$ is consistent, it suffices, from diagram (2), to show that $L(P_1)$ and $L(P_2)$ are consistent at Q_3 . This follows from the fact that X_i has at most rank 1 at Q_3 . Similarly, $\{N(\mathcal{M})\}$ is consistent. Further, since, at any minimal prime Q , $(X_i)_Q = Y_Q$ and $Z_Q = 0$ or vice versa, we have $(X_i)_{\mathcal{M}} = L(\mathcal{M}) \oplus N(\mathcal{M})$ for all maximal ideals \mathcal{M} . Hence, by the Consistency Theorem, there exist lattices L and N such that $X_i = L \oplus N$; yet X_i is indecomposable so either $L = 0$ or $N = 0$ and so either $Y = 0$ or $Z = 0$. Thus, $S^{-1}X_i$ is indecomposable.

As a result, localizing equation (8) by S^{-1} gives

$$S^{-1}X_4^{(a_4)} \oplus \cdots \oplus S^{-1}X_t^{(a_t)} = S^{-1}M' \simeq S^{-1}X_4^{(b_4)} \oplus \cdots \oplus S^{-1}X_t^{(b_t)}$$

where each $S^{-1}X_i$ is non-zero and indecomposable over $S^{-1}R$. Since $S^{-1}R$ has one less maximal ideal than R , by induction

$$\sum_{i=4}^t a_i = \sum_{i=4}^t b_i.$$

This finishes the proof of the implication (B) \Rightarrow (C).

(C) \Rightarrow (A). Suppose that X is an indecomposable R -lattice and that for some maximal ideal P of R , X_P decomposes as $X_P = Y \oplus Z$. Of course, if Q is a minimal prime $Q \subset P$ such that $X_Q \neq 0$, then either

$$Y_Q = X_Q \neq 0 \text{ and } Z_Q = 0 \text{ or vice versa} \quad (12)$$

since X is isomorphic to an ideal. This will yield a contradiction to (C).

Note that the graph of R has a loop. To see this, suppose it does not and let

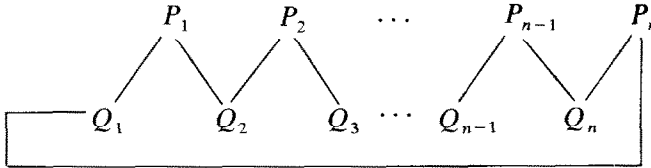
$$S_L = \{\mathcal{M} \in \text{Maxspec } R \mid \text{there exists a path in } \text{gph}(R) \text{ from } \mathcal{M}, \text{ not passing through } P, \text{ to a minimal prime } Q \subset P \text{ such that } Y_Q \neq 0\}$$

and let

$$S_N = \{\mathcal{M} \in \text{Maxspec } R \mid \text{there exists a path in } \text{gph}(R) \text{ from } \mathcal{M}, \text{ not passing through } P, \text{ to a minimal prime } Q \subset P \text{ such that } Z_Q \neq 0\}.$$

Set $L(\mathcal{M}) =: Y$ for $\mathcal{M} = P$, $=: X_{\mathcal{M}}$ for $\mathcal{M} \in S_L$, and $=: 0$ otherwise. Similarly, set $N(\mathcal{M}) =: Z$ for $\mathcal{M} = P$, $=: X_{\mathcal{M}}$ for $\mathcal{M} \in S_N$, and $=: 0$ otherwise. The consistency of $\{L(\mathcal{M})\}$ and $\{N(\mathcal{M})\}$ results from the fact that the localizations of X form a consistent family, that X has at most rank 1 at any minimal prime and that $\text{gph}(R)$ has no loops by assumption. In addition, by (12) and the ‘no loop hypothesis’, S_N and S_L are disjoint and so $X_{\mathcal{M}} = L(\mathcal{M}) \oplus N(\mathcal{M})$ for every maximal ideal \mathcal{M} . Consequently, by the Consistency Theorem, there exist lattices L and N such that $X = L \oplus N$. This contradicts the indecomposability of X and so $\text{gph}(R)$ must have a loop.

Since R is ΣI , the maximal ideals of the loop each contain exactly two minimal prime ideals. By appropriate localizations, assume $\text{gph}(R)$ has the form



To find the contradiction, set $S_i = R/(Q_i \cap Q_{i+1})$ and $R_i = R/Q_i$ where $n+1$ denotes 1. Each S_i and R_i is an indecomposable R -lattice since each is a local ring. Consider the R -lattices $M = S_1 \oplus S_2 \oplus \cdots \oplus S_n$ and $N = R \oplus R_1 \oplus \cdots \oplus R_n$. It is easy to check that

$$M_{P_i} \simeq (S_i)_{P_i} \oplus (R_i)_{P_i} \oplus (R_{i+1})_{P_i} \simeq R_{P_i} \oplus (R_i)_{P_i} \oplus (R_{i+1})_{P_i} \simeq N_{P_i}$$

for every i such that $1 \leq i \leq n$. Since R is a semilocal ring, $M \simeq N$ and thus, $S_1 \oplus S_2 \oplus \cdots \oplus S_n \simeq R \oplus R_1 \oplus \cdots \oplus R_n$. Yet each S_i , R_i and R is indecomposable so that M does not have a unique number of indecomposable summands. This contradicts (C) and so finishes the implication (C) \Rightarrow (A). This proves the theorem. \square

Example 4.7. (1) The ring $\mathbb{Z}G_4$, where G_4 is a cyclic group of order 4, is a ΣI ring-order with K-S-A and UNIS since $\text{Pic } \mathbb{Z}G_4 = 1$ and there is exactly one singular maximal ideal.

(2) The ring $\mathbb{Z}G_6$ has a loop in its graph and so $\mathbb{Z}G_6$ does not have UNIS nor K-S-A.

Torsionfree cancellation is the property that every lattice can be cancelled from a direct sum isomorphism; that is, every X satisfies

$$M \oplus X \simeq N \oplus X \quad \text{imply} \quad M \simeq N. \quad (\text{A}')$$

Wiegand, in [5, 2.7], proves that a Bass ring R has torsionfree cancellation if and only if $D(R) = 0$. This is also true for ΣI rings.

Theorem 4.8. *Let R be a ΣI ring. Then R has torsionfree cancellation if and only if $D(R) = 0$.*

Proof. ‘ \Rightarrow ’. For any $u \in G =: (\tilde{R}/c)^*$, $\tilde{R}^u \simeq \tilde{R}$ since $\Delta_{\tilde{R}} = (\tilde{R}/c)^*$. Thus, $R^u \oplus \tilde{R} \simeq (R \oplus \tilde{R})^u \simeq R \oplus \tilde{R}^u \simeq R \oplus \tilde{R}$. But by hypothesis, if R has torsionfree cancellation, then $R^u \simeq R$ for every $u \in G$. This implies that $D(R) = \{[R^u] \mid u \in G\} = 0$.

‘ \Leftarrow ’. If $D(R) = 0$, then, by Theorem 3.7, $D(M) = 0$. On the other hand, if $M \oplus X \simeq N \oplus X$, then by [6, 1.9], $N \simeq M^u$ for some $u \in (\tilde{R}/c)^* =: G$. Yet $0 = D(M) = \{[M^u] \mid u \in G\}$ implies that $N \simeq M^u \simeq M$ so R has torsionfree cancellation. \square

Thus, direct sum cancellation of every R -lattice X , where R is a ΣI ring, holds rarely and so the next question to ask is what conditions on the lattice X are sufficient so that implication (A') holds?

Levy and Wiegand, in [3, 6.2 and 6.3], provide some sufficient conditions on X for Bass rings; for example, if X is projective, then X can be cancelled. However, as the following example demonstrates, this condition is insufficient when the ring is ΣI but not Bass.

Example 4.9. Let R be the triad of 3 copies of the integers \mathbb{Z} pulled back over the field $k = \mathbb{Z}/5\mathbb{Z}$ (the integers modulo 5). That is,

$$R = \{(x_1, x_2, x_3) \in \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \mid x_1 \equiv x_2 \equiv x_3 \text{ modulo } 5\}.$$

Let M be the R -lattice $M = R \cdot (1, 1, 0) \oplus R \cdot (0, 1, 1)$. Then there exists an element $u \in (\tilde{R}/c)^* =: G$ such that $M^u \oplus R \simeq M \oplus R$ but $M \not\simeq M^u$.

Proof. First of all, notice that $G = (\tilde{R}/c)^* = \mathbb{Z}_5^* \times \mathbb{Z}_5^* \times \mathbb{Z}_5^*$, where \mathbb{Z}_5 denotes the units of $\mathbb{Z}/5\mathbb{Z}$. As a result, the liftable units form the non-trivial subgroup $\Lambda_R = H \times H \times H$ where H is the subgroup of \mathbb{Z}_5^* consisting of the elements $\bar{1}$ and $\bar{4}$. It is easily checked that $\Delta_R = \{(x, x, x) \mid x \in \mathbb{Z}_5^*\}$ and $\Delta_M = \{(x, xy, y) \mid x, y \in \mathbb{Z}_5^*\}$.

Set $u = (2, 2, 2) \in G$ and observe that $M^u \oplus R \simeq (M \oplus R)^u \simeq M \oplus R^u \simeq M \oplus R$ since M and R are both faithful and $u \in \Delta_R$. On the other hand, $M^u \not\simeq M$ because $u \notin \Delta_M \cdot \Lambda_R$ by [6, 1.6]. To see this, suppose that $u \in \Delta_M \cdot \Lambda_R$. Then $u = (\bar{2}, \bar{2}, \bar{2}) = (x, xy, y) \cdot (\lambda_1, \lambda_2, \lambda_3)$ where $(\lambda_1, \lambda_2, \lambda_3) \in \Lambda_R$. A quick computation shows that $\bar{2} = \lambda_1 \cdot \lambda_2^{-1} \cdot \lambda_3 \in H$ which is impossible according to the definition of H . Hence, $M^u \not\simeq M$. \square

Levy and Wiegand [3, 6.6] also considered the topic of power-cancellation,

which is the implication

$$M \oplus X \simeq N \oplus X \Rightarrow M^{(m)} \simeq N^{(m)} \quad \text{for some } m. \quad (\text{B}')$$

For Bass rings, the exponent of $\text{Pic } R$ plays an important role in the power cancellation behavior. The same phenomenon occurs for ΣI rings as well.

Theorem 4.10 (Power Cancellation). *Let R be a ΣI ring. Then*

- (i) *If $D(R)$ is a torsion group with finite exponent x , then x is the least integer m satisfying (B');*
- (ii) *If $D(R)$ is a torsion group with infinite exponent, then (B') is true, but no fixed value of m works for all M, N, X ; and*
- (iii) *If $D(R)$ has elements of infinite order, then there are R -lattices M, N , and X such that (B') fails for all m .*

Proof. (i) Since $D(M)$ is a homomorphic image of $\Pi' D(R)$ by Theorem 3.7 and since $x = \text{exponent}(D(R)) = \text{exponent}(\Pi' D(R))$, we have that x is divisible by $\text{exponent}(D(M))$. Now if $M \oplus X \simeq N \oplus X$, then $M^u \simeq N$ by [5, 2.3]. But $u^x = 1$ in $D(M)$ and so $N^{(x)} \simeq (M^u)^{(x)} \simeq (M^{(x)})^{u^x} \simeq M^{(x)}$.

To see that x is the least such integer, note that since $D(\tilde{R}) = (\tilde{R}/c)^*$, then for any $u \in (\tilde{R}/c)^*$, $R^u \oplus \tilde{R} \simeq (R \oplus \tilde{R})^u \simeq R \oplus \tilde{R}^u \simeq R \oplus \tilde{R}$. But if z is an integer that satisfies (B'), then $(R^u)^{(z)} \simeq R^{(z)}$ and so $u^z = 1$ in $D(R^{(z)}) \simeq D(R)$ for all $u \in (\tilde{R}/c)^*$. This occurs if and only if $x = \text{exponent}(D(R))$ divides z . Hence, x is the least such integer.

(ii) This is proven similarly to (i).

(iii) If $u \in D(R)$ has infinite order, then $u^z \neq 1$ for all positive integers z . Hence $(R^u)^{(z)} \not\simeq R$ for all positive integers z . Yet as above, $R^u \oplus \tilde{R} \simeq R \oplus \tilde{R}$. \square

Determining the size of a given genus is another classical direct sum type question. Levy and Wiegand show that, for Bass rings, if $\text{Pic } R$ is a finite group, then the order of $G(M)$ divides the order of $\text{Pic } R$, [3, 5.4]. This is because $G(M)$ is always a homomorphic image of $\text{Pic } R$. However, this may not be true for every ΣI ring; nonetheless, there is an analogous result for ΣI rings. Let $|G|$ denote the order of any group G .

Theorem 4.11. *Let R be a ΣI ring with $\text{Pic } R$ finite. Let t be the number of indecomposable genera as in Lemma 3.3. Then, for every R -lattice M , $|G(M)|$ divides $|\text{Pic } R|^t$.*

Proof. By Theorem 3.7, $G(M)$ is a homomorphic image of $\Pi' \text{Pic } R$. \square

Of course, $\text{Pic } R$ need not be finite; in which case, it still may be a torsion group with a finite exponent x . This positive integer determines a bound on the genus

exponent of an R -lattice M , which is the exponent of the genus class group $G(M)$; that is, the genus exponent is the least positive integer m such that if

$$N \in G(M) \Rightarrow M^{(m)} \simeq N^{(m)}. \quad (C')$$

Levy and Wiegand derive a bound, namely the exponent of $\text{Pic } R$, for all genera exponents where R is a Bass ring [3, 5.6]. This same bound also works for ΣI rings. Note that the exponent of $\text{Pic } R$ is *exactly* the genus exponent of $G(R)$ since $\text{Pic } R = G(R)$.

Theorem 4.12 (Genus Exponent). *Let R be a ΣI ring. Then*

- (i) *If $\text{Pic } R$ is a torsion group with finite exponent x , then (C') holds for $m = x$ but fails for $m < x$;*
- (ii) *If $\text{Pic } R$ is a torsion group with infinite exponent, then (C') holds, but no fixed value of m works for all M and N ; and*
- (iii) *If $\text{Pic } R$ contains an element of infinite order, then (C') is false.*

Proof. (i) To see that x works in (C') , note that $x = \exp(\text{Pic } R) = \exp(\prod^t \text{Pic } R)$ is divisible by $\exp(G(M)) = y$, say $x = y \cdot z$, by Theorem 3.7. Hence, given $[N]$ and $[M] \in G(M) = G(N)$, we have $x \cdot [N] = y \cdot z \cdot [N] = [M] = x \cdot [M]$ or $N^{(x)} \simeq M^{(x)}$. On the other hand, x is the least such integer that works for all genera for if $[I] \in \text{Pic } R$ such that $I^{(m)} \simeq R^{(m)}$ for some m , then $I \cdot I \cdot \dots \cdot I$ (m factors) $= R \cdot R \cdot \dots \cdot R$ (m factors) $= R$ and so the order of $[I]$ in $\text{Pic } R$ divides m . Of course, x is the least integer such that the order of any element in $\text{Pic } R$ divides x .

(ii) This is shown similarly to (i).

(iii) Suppose $[I] \in \text{Pic } R$ has infinite order. Then $[I]^m \neq [R]^m = [R]$ for all positive integers m . Since $\text{Pic } R$ can be thought of as both an additive or a multiplicative group, $[I]^m = m \cdot [I]$ and so $I^{(m)} \cong R^{(m)}$ or (C') fails. \square

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